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On the Fundamental System of Neighborhoods of a Subspace in a Complex Space

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Abstract. Let X be a complex space (or complex manifold) and let Y be a closed subspace (or closed submanifold) of X. The purpose of this paper is to show the existence of a fundamental system of weakly 1-complete (or holomorphically convex or Stein) neighborhoods for each level set Y_c in X under the condition that the restriction $[\widetilde{Y}]|_{\widetilde{Y}}$ of the corresponding line bundle $[\widetilde{Y}]$ to $\widetilde{Y}$ is negative, where $\widetilde{Y}$ is given by the monoidal transformation of X with center Y.

Introduction

Let X be a complex space and let Y be a subspace of X. Then Y.-T. Siu and M. Schneider showed the following.

Suppose Y is a closed Stein subspace of X. Then every compact subset of Y admits a fundamental system of Stein neighborhoods in X.

In [14] Siu showed the above theorem by using Richberg's result ([11], Satz 3.3) and the techniques of generalized normal bundle in the sense of Grauert ([3], pp. 351-353). Further he showed in the same paper that every Stein subvariety admits a Stein neighborhood in a complex space. On the other hand, in [12] Schneider showed the above theorem by using Richberg's result and the techniques of the monoidal transformation of X with center Y.

In this paper we shall show that if Y is a closed subspace (or closed submanifold) of a complex space (or complex manifold) X, there exists a fundamental system of weakly 1-complete neighborhoods for each level set Y_c in X since Y is regarded as a weakly 1-complete space under some assumptions (2). In particular, if Y is a closed Stein subspace, we shall show the existence of a fundamental system of Stein neighborhoods for each level set Y_c in X (3, Theorem 3.1). Moreover in the case where Y is a holomorphically convex subspace, we shall also show the existence of a fundamental system of holomorphically convex neighborhoods for the level set Y_c in X under some assumptions (3, Theorem 3.3).

For these situations, the negativity of the restriction $[\widetilde{Y}]|_{\widetilde{Y}}$ of $[\widetilde{Y}]$ to $\widetilde{Y}$ is the important condition, where $\widetilde{Y}$ is given by the monoidal transformation of X with center Y (see 1, (1.5)).

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1. Preliminaries and the Fundamental Lemmas

(1.1) Through this paper, we suppose that complex spaces are all reduced and have the countable topology and that complex manifolds are always paracompact. Let $\boldsymbol{R}$, $\boldsymbol{R}^+$ and $\boldsymbol{C}$ be real, positive real and complex number field respectively. Let X be a complex space. When p is a strictly plurisubharmonic (or plurisubharmonic) C^∞-function on X, for simplicity, we write that p is C^∞-spsh (or C^∞-psh) on X.

(1.2) Let B be a holomorphic line bundle on a complex space X. If $\{f_{\alpha\beta}\}$ is the system of transition functions for B with respect to some open covering $\{U_\alpha\}$ of X, a metric on B with respect to this covering is given by the system $\{h_\alpha\}$ of positive C^∞-functions defined on U_α, such that $h_\beta = |f_{\alpha\beta}|^2 h_\alpha$ on $U_\alpha \cap U_\beta$.

Definition 1.1. (negative line bundle) A line bundle B on X is said to be *negative* (or *positive*) if for a suitable choice of $\{f_{\alpha\beta}\}$ and $\{U_\alpha\}$ as above, there exists a metric $\{h_\alpha\}$ with respect to $\{U_\alpha\}$ such that $\log h_\alpha$ (or $-\log h_\alpha$) is C^∞-spsh on U_α for every α.

(1.3) Definition 1.2. ([7], weakly 1-complete) A complex space X is said to be a *weakly* 1-*complete* space with an exhaustion function Ψ if there exists a C^∞-psh function Ψ on X such that for every $c \in \boldsymbol{R}$ the level set $X_c := \{x \in X\,;\, \Psi(x) < c\} \Subset X$, where $X_c \Subset X$ means that $\overline{X}_c$ is compact and contained in the interior of X. We may assume that $\Psi > 0$, since we can take $\exp(\Psi)$ instead of Ψ.

Remark 1. (1) Let X and Y be complex spaces. If $\pi: X \to Y$ is a proper holomorphic map and if Y is a weakly 1-complete space with an exhaustion function Φ, then X becomes a weakly 1-complete complex space with an exhaustion function $\pi^*\Phi$. In particular any closed subspace of a weakly 1-complete complex space is naturally weakly 1-complete.
(2) We assume that X is holomorphically convex complex space. Then we can regard that X is weakly 1-complete. Now we shall show this. By Remmert quotients, X admits a proper surjective holomorphic mapping $\pi: X \to S$, where S is a Stein space. On the other hand, by the embedding theorem, there exists a proper holomorphic mapping τ of S into some complex number space C^N. Let $(z_1,\cdots, z_n)$ be coordinates of a point in C^N. We put $\Psi = (\tau \circ \pi)^* (\sum_{i=1}^{N} |z_i|^2)$. Since $\sum_{i=1}^{N} |z_i|^2$ is C^∞-spsh on C^N, Ψ is C^∞-psh on X. Since $\tau \circ \pi$ is proper, X is a weakly 1-complete space with an exhaustion

function Ψ.

In this paper, when a holomorphically convex space X is given, we regard it as a weakly 1-complete space by the above remark. Hence we use the notation X_c, $c \in \boldsymbol{R}$, for the level set with an exhaustion function Ψ as above.

(1.4) There is the following vanishing theorem F on weakly 1-complete space. This was proved by A. Fujiki, who extended a Nakano's result for more general case ([7], p. 169, Theorem 2).

THEOREM F. ([2], p. 475, Theorem N′) *Let X be a weakly 1-complete complex space. Let $\mathscr{F}$ be a coherent analytic sheaf on X and let L be a positive line bundle on X. Then for every $c \in \boldsymbol{R}$, there exists a positive integer n_0 such that $H^i(X_c, \mathscr{F} \otimes \mathcal{O}(L^{\otimes n}) \otimes \mathcal{O}(E)) = 0$ for all $i \geqq 1$, $n \geqq n_0$, and for every semipositive line bundle E on X.*

Theorem F is used in 3, Theorem 3.3.

(1.5) Now, for the pair of a complex space X and a closed subspace Y of X, there exists the monoidal transformation of X with center Y such that the holomorphic map $\pi: \widetilde{X} \to X$ is proper and $\pi: \widetilde{X} \setminus \widetilde{Y} \xrightarrow{\sim} X \setminus Y$ is biholomorphic where $\widetilde{X}$ is a complex space and $\widetilde{Y} = \pi^{-1}(Y)$ (see [5], pp. 315–317). Then $\widetilde{Y}$ is an effective cartier divisor (i.e., at each point of $\widetilde{X}$ the ideal sheaf $\mathscr{I}_{\widetilde{Y}}$ of $\widetilde{Y}$ in $\widetilde{X}$ is generated by a single element which is not a zero divisor) on complex space $\widetilde{X}$. In the case where X is a complex manifold and Y is a closed submanifold of X, $\widetilde{Y}$ is non-singular divisor on complex manifold $\widetilde{X}$.

(1.6) In general, we take the corresponding line bundle $[Y]$, where Y is an effective cartier divisor (or non-singular divisor) on complex space (or complex manifold) X. We denote by $[Y]|_Y$ the restriction of $[Y]$ to Y. The restriction means the analytic restriction, that is $[Y]|_Y = i^*[Y]$, where $i: Y \hookrightarrow X$ is the inclusion map.

Under the assumption of the negativity of $[Y]|_Y$, we have the following two fundamental lemmas due to Cornalba and Fujiki.

CORNALBA'S LEMMA. ([1], pp. 232–233) *Let X be a complex manifold and let Y be non-singular divisor on X with a proper and smooth holomorphic map of Y onto a manifold Z. Let $[Y]$ be the corresponding line bundle. Assume that the restriction $[Y]|_Y$* of *$[Y]$ to Y is negative. Then there exists a non-negative C^∞-psh function Ψ* on *X* which is positive *C^∞-spsh on $X - Y$ and vanishes only on Y.*

FUJIKI'S LEMMA. ([2], p. 483, Lemma 4 and its proof) *Let X be a complex space and let Y be an effective cartier divisor on X. Let $[Y]$ be the corresponding line bundle. Assume that the following conditions are satisfied*:

(1) *Y is weakly 1-complete,*

(2) *the restriction $[Y]|_Y$ of $[Y]$ to Y is negative.*

Then for every $c \in \boldsymbol{R}$ there exists a neighborhood U_c of Y_c in X with $U_c \cap Y = Y_c$ which has the following properties:

(i) $[Y]$ *is negative on* U_c,

(ii) *there exists a non-negative* C^∞*-psh function* Ψ *on* U_c *which is positive* C^∞*-spsh on* $U_c - Y_c$ *and vanishes only on* Y_c.

(1.7) From Cornalba's Lemma and Fujiki's Lemma, we obtain the following two lemmas.

LEMMA 1.1. *Let* X *be a complex manifold and let* Y *be a closed submanifold of* X. *Let* $[\widetilde{Y}]$ *be the line bundle corresponding to the non-singular divisor* $\widetilde{Y}$. *Assume that the restriction* $[\widetilde{Y}]|_{\widetilde{Y}}$ *of* $[\widetilde{Y}]$ *to* $\widetilde{Y}$ *is negative. Then there exist a neighborhood* U *of* Y *in* X *and a non-negative* C^∞*-psh function* φ *on* U *which is positive* C^∞*-spsh on* $U-Y$ *and vanishes only on* Y.

PROOF. By Cornalba's Lemma, there exist a neighborhood $\widetilde{U}$ of $\widetilde{Y}$ in $\widetilde{X}$ and a non-negative C^∞-psh $\tilde{\varphi}$ on $\widetilde{U}$ which is positive C^∞-spsh on $\widetilde{U}-\widetilde{Y}$ and vanishes only on $\widetilde{Y}$. Since $\pi\colon \widetilde{X}\setminus\widetilde{Y} \rightrightarrows X\setminus Y$ is biholomorphic, we put $\varphi = \tilde{\varphi}\circ\pi^{-1}$ on $U-Y$ where $U = \pi(\widetilde{U})$. Then we can define that $\varphi=0$ on Y. φ is a C^∞-spsh function on $U-Y$. Hence we obtain the conclusion. q.e.d.

REMARK 2. Let X be a complex manifold and let Y be a closed Stein submanifold of X. Then the restriction $[\widetilde{Y}]|_{\widetilde{Y}}$ of $[\widetilde{Y}]$ to $\widetilde{Y}$ is always negative.

LEMMA 1.2. *Let* X *be a complex space and let* Y *be a closed weakly* 1*-complete subspace of* X. *Let* $[\widetilde{Y}]$ *be the line bundle corresponding to the effective cartier divisor* $\widetilde{Y}$. *Assume that the restriction* $[\widetilde{Y}]|_{\widetilde{Y}}$ *of* $[\widetilde{Y}]$ *to* $\widetilde{Y}$ *is negative. Then for every* $c \in \boldsymbol{R}$ *there exists a neighborhood* U_c *of* Y_c *in* X *with* $U_c \cap Y = Y_c$ *which has the following property*: *there exists a non-negative* C^∞*-psh function* φ *on* U_c *which is positive* C^∞*-spsh on* $U_c - Y_c$ *and vanishes only on* Y_c.

PROOF. we put $\widetilde{Y}_c = \pi^{-1}(Y_c)$ for each level set Y_c. For $\widetilde{Y}_c$, we can apply Fujiki's Lemma. Then there exist a neighborhood $\widetilde{U}_c$ of $\widetilde{Y}_c$ in X with $\widetilde{U}_c \cap \widetilde{Y} = \widetilde{Y}_c$ and a C^∞-psh $\tilde{\varphi}$ on $\widetilde{U}_c$ which is postive C^∞-spsh on $\widetilde{U}_c - \widetilde{Y}_c$ and vanishes only on $\widetilde{Y}_c$. We put $U_c = \pi(\widetilde{U}_c)$. Then since $\pi\colon \widetilde{X}\setminus\widetilde{Y} \rightrightarrows X\setminus Y$ is biholomorphic, as in the proof of Lemma 1.1, there exist a neighborhood U_c of Y_c in X with $U_c \cap Y = Y_c$ for every $c \in \boldsymbol{R}$ and a C^∞-psh function φ on U_c. This φ is a desired function. q.e.d.

(1.8) The following lemma will be used in 2, (2.4).

LEMMA 1.3. *Let* X *be a complex manifold and let* Y *be a non-singular divisor. Let*

$[Y]$ be the corresponding line bundle.

(1) *Assume that $[Y]$ is negative on X. Then there exists a metric function φ along the fibre of $[Y]$ which is positive C^∞-spsh on $[Y]-X$ and vanishes only on X, where X is regarded as the zero section of X in $[Y]$.*

(2) *Assume that the restriction $[Y]|_Y$ of $[Y]$ to Y is negative. Then there exists a metric function φ along the fibre of $[Y]|_Y$ which is positive C^∞-spsh on $[Y]|_Y - Y$ and vanishes only on Y, where Y is regarded as the zero section of Y in $[Y]|_Y$.*
(For the construction of the desired metric function, see [17], pp. 7–8 and [15], p. 13)

REMARK 3. Let E be a holomorphic vector bundle on X. Then, if E is negative in the sense of Nakano ([7], p. 175, 2.4), its metric function along the fibre of E satisfies the same conclusion as Lemma 1.3 (1) ([17], pp. 7–8).

2. Negativity of the Line Bundle Corresponding to Non-Singular Divisor and Effective Cartier Divisor

(2.1) The following lemma will play an important role for the construction of fundamental system of weakly 1-complete neighborhoods of each level set Y_c of a weakly 1-complete subspace (or submanifold) in a complex space (or manifold).

LEMMA 2.0. *Let X be a complex space and let Y be a weakly 1-complete subspace with a positive exhaustion function of X. Assume that the following conditions are satisfied:*

(1) *there exist a neighborhood U_c of each level set Y_c in X and a non-negative C^∞-psh function Ψ_c on U_c such that $Y_c=\{y\in Y;\ \Psi_c(y)<c\}$ and $U_c\cap Y=Y_c$ for every $c\in \boldsymbol{R}^+$,*

(2) *there exist a neighborhood V_c of each level set Y_c in X and a non-negative C^∞-psh function φ_c on V_c which is C^∞-spsh on V_c-Y_c and vanishes only on Y_c, where $V_c\cap Y=Y_c$ for every $c\in\boldsymbol{R}^+$.*

Then there exists a weakly 1-complete neighborhood W_c of each level set Y_c in X with $W_c\cap Y=Y_c$ for every $c\in\boldsymbol{R}^+$.

PROOF. Now fix an arbitrary $c\in\boldsymbol{R}^+$ and take $d_1\in\boldsymbol{R}^+$ such that $d_1>c$. We put $W=U_{d_1}\cap V_c$. Then by taking sufficiently small neighborhoods U_{d_1} and V_c of Y_c in X, without loss of generality we may assume that W is relatively compact in X. Hence for any $d_2<d_1$ and for a sufficiently small $\varepsilon>0$, the subset W_ε of W defined by $W_\varepsilon=\{x\in W;\ \Psi_c(x)<d_2,\ \varphi_c(x)<\varepsilon\}$ is relatively compact in W. Put $\Phi_c=\frac{c}{\varepsilon}\varphi_c+\Psi_c$, where ε is suitably chosen for c. Then the subset $W_c=\{x\in W;\ \Phi_c(x)<c\}$ of W is relatively compact. Hence W_c is a weakly 1-complete subspace with an exhaustion function $(c-\Phi_c)^{-1}$ and $W_c\cap Y=Y_c$ from that $\varphi_c=0$ on Y_c. From the construction of Φ_c, for a sufficiently small ε we can obtain arbitrarily small W_c. q.e.d.

(2.2) Here we shall treat the case where a complex space (or complex manifold) is weakly 1-complete. Let X be a weakly 1-complete complex space (or manifold) with an exhaustion function Ψ and let Y be a closed subspace (or submanifold). Then we can consider that Y is weakly 1-complete (see 1, (1.3), Remark 1, (1)). For, each level set Y_c becomes weakly 1-complete with an exhaustion function $\widetilde{\Psi}$, since Y_c is relatively compact in Y for every $c \in \boldsymbol{R}$, where $\widetilde{\Psi} = \Psi|_Y$ and $Y_c = \{y \in Y\,;\, \widetilde{\Psi}(y) < c\}$. Then $X_c \cap Y = Y_c$ for every $c \in \boldsymbol{R}$ where $X_c = \{x \in X\,;\, \Psi(x) < c\}$.

In the following, we can assume without loss of generality that an exhaustion function is positive (see 1, (1.3)).

First we consider the case where X is a weakly 1-complete manifold.

THEOREM 2.1. *Let X be a weakly 1-complete manifold with an exhaustion function Ψ and let Y be a closed submanifold. Let $[\widetilde{Y}]$ be the line bundle corresponding to the non-singular divisor $\widetilde{Y}$. Assume that the restriction $[\widetilde{Y}]|_{\widetilde{Y}}$ of $[\widetilde{Y}]$ to $\widetilde{Y}$ is negative. Then for every $c \in \boldsymbol{R}^+$ there exists a sufficiently small weakly 1-complete neighborhood U_c of Y_c in X with $U_c \cap Y = Y_c$.*

PROOF. By Lemma 1.1, there exist a neighborhood U of Y in X and a C^∞-psh function φ on U which is C^∞-spsh on $U - Y$ and vanishes only on Y. Then $X_c \cap U$ can be regarded as the neighborhood of Y_c in X. We put $\varphi_1 = \varphi|_{X_c \cap U}$ which is the restriction of φ to $X_c \cap U$. Since we can apply Lemma 2.0 for C^∞-psh functions Ψ and φ_1, there exists a sufficiently small weakly 1-complete neighborhood U_c of Y_c in X with $U_c \cap Y = Y_c$. q. e. d.

Next for the case of a weakly 1-complete space, we obtain the following

THEOREM 2.2. *Let X be a weakly 1-complete complex space with an exhaustion function Ψ and let Y be a closed subspace. Let $[\widetilde{Y}]$ be the line bundle corresponding to the effective cartier divisor $\widetilde{Y}$. Assume that the restriction $[\widetilde{Y}]|_{\widetilde{Y}}$ of $[\widetilde{Y}]$ to $\widetilde{Y}$ is negative. Then for every $c \in \boldsymbol{R}^+$ there exists a sufficiently small weakly 1-complete neighborhood U_c of Y_c in X with $U_c \cap Y = Y_c$.*

PROOF. Since Y is a weakly 1-complete subspace, we can apply Lemma 1.2 to Y. Therefore there exist a neighborhood W_c of Y_c in X and C^∞-psh function φ on W_c which is C^∞-spsh on $W_c - Y_c$ and vanishes only on Y_c for every $c \in \boldsymbol{R}^+$. Then as in the proof of Theorem 2.1, we have the conclusion. q. e. d.

(2.3) For the case of complex manifold, we shall consider the application of Grauert's method [3].

THEOREM 2.3. *Let X be a complex manifold and let Y be a compact closed submanifold. Let $[\widetilde{Y}]$ be the line bundle corresponding to the compact non-singular divisor $\widetilde{Y}$. Assume* *that the restriction* $[\widetilde{Y}]|_{\widetilde{Y}}$ *of* $[\widetilde{Y}]$ *to* $\widetilde{Y}$ *is negative. Then there exists a holomorphically convex neighborhood of* Y *in* X.

PROOF. Since $\widetilde{Y}$ is compact non-singular divisor and $[\widetilde{Y}]|_{\widetilde{Y}}$ is weakly negative in the sense of Grauert ([3], p. 341, Satz 1), there exists a holomorphically convex neighborhood $\widetilde{U}$ of $\widetilde{Y}$ in $\widetilde{X}$ ([3], p. 353, Satz 8 and [8], p. 195, Theorem I). On the other hand, from the property of the monoidal transformation, we take $\pi_*(\mathcal{O}_{\widetilde{U}}) = \mathcal{O}_U$ where $U = \pi(\widetilde{U})$ and $\pi: \widetilde{X} \to X$. From this, U is a holomorphically convex neighborhood of Y in X. q. e. d.

(2.4) Here, for the complex manifold case, we treat the application of the generalized normal bundle in the sense of Grauert ([3], pp. 351–353) and a 1-convex holomorphic map ([6], [13]).

DEFINITION 2.1. ([13], 1–convex holomorphic map) Let $\tau: X \to S$ be a holomorphic map of complex manifolds. Then τ is said to be 1-*convex* if there exists a real valued exhaustion C^∞-function $\Psi: X \to (-\infty, c_*)$ such that the restriction of τ to $\{\Psi \leqq c\}$ is proper for every $-\infty < c < c_*$ and Ψ is C^∞-spsh on $\{\Psi > c_\#\}$ for some $-\infty < c_\# < c_*$, where $c_* \in \boldsymbol{R} \cup \{\infty\}$.

THEOREM 2.4. *Let* X *be a complex manifold and let* Y *be a closed submanifold of* X. *Let* $[\widetilde{Y}]$ *be the line bundle corresponding to the non-singular divisor* $\widetilde{Y}$. *Assume that the following conditions are satisfied*:

(1) *the restriction* $[\widetilde{Y}]|_{\widetilde{Y}}$ *of* $[\widetilde{Y}]$ *to* Y *is negative,*

(2) *there exists a holomorphic map* τ *of* X *into a Stein manifold* S *which has a* C^∞-*spsh exhaustion function* Ψ,

(3) *the restriction of* τ *to* Y *is proper.*

Then for every $c \in \boldsymbol{R}$ *there exists a holomorphically convex neighborhood* U_c *of* Y_c *in* X *with* $U_c \cap Y = Y_c$, *where* $Y_c = \{y \in Y;\ \tau^*\Psi(y) < c\}$. *In particular, no positive dimensional compact irreducible analytic subset of* U_c *intersects with* $U_c - Y_c$ *for every* $c \in \boldsymbol{R}$.

PROOF. Let $\pi: \widetilde{X} \to X$ be the monoidal transformation of X with center Y. We put $\widetilde{Y}_c = \pi^{-1}(Y_c) = \pi^{-1}(Y \cap \tau^{-1}(S_c))$ where S_c is a Stein manifold. The restriction of $\tau \circ \pi$ to $\widetilde{Y}$ is proper holomorphic map. On the other hand, by Lemma 1.3 (2), for the non-singular divisor $\widetilde{Y}$, there exists a metric function $\widetilde{\varphi}$ along the fibre of $[\widetilde{Y}]|_{\widetilde{Y}}$ which is C^∞-spsh on $[\widetilde{Y}]|_{\widetilde{Y}} - \widetilde{Y}$. Now, we identify $[\widetilde{Y}]|_{\widetilde{Y}}$ with the generalized normal bundle on $\widetilde{Y}$ of the linear space L in the sense of Grauert ([3], pp. 351–353). By applying Richberg's Theorem ([11], Satz 3.3) to $\widetilde{\varphi}$, we can take a relatively compact neighborhood W of the zero section $O_{\widetilde{Y}_c}$ of $\widetilde{Y}_c$ in L such that $W \cap \widetilde{Y} = \widetilde{Y}_c$, and there exists a C^∞-psh function φ on W which is C^∞-spsh on $W - \widetilde{Y}_c$ and $\varphi|_{[\widetilde{Y}]|\widetilde{Y}_c - \widetilde{Y}_c} = \widetilde{\varphi}$, where we can consider $\widetilde{Y}_c$ the zero section of $\widetilde{Y}_c$ in $[\widetilde{Y}]|_{\widetilde{Y}_c}$. Again, we take a relatively compact neighborhood W_1 of $O_{\widetilde{Y}_c}$ in L such that $W_1 \cap [\widetilde{Y}]|_{\widetilde{Y}_c} \Subset W$. Then for $\varepsilon > 0$, we can choose a neighborhood W_2 of $O_{\widetilde{Y}_c}$ in L such that $W_2 \Subset W_1$, $\varphi(x) > \varepsilon$ on $\partial W_1 \cap [\widetilde{Y}]|_{\widetilde{Y}_c}$ and $\varphi(x) < \varepsilon$ on $\partial W_2 \cap [\widetilde{Y}]|_{\widetilde{Y}_c}$. On the other hand, since $\tau \circ \pi (\widetilde{Y})$ is a closed analytic subset in S by the proper mapping theorem ([4], p. 162, Theorem 5), $\tau \circ \pi (\widetilde{Y})$ is Stein analytic subset in S. In particular, since $\tau \circ \pi (\widetilde{Y}_c) = \tau \circ \pi (\widetilde{Y}) \cap S_c$ and S_c is Stein manifold, $\tau \circ \pi (\widetilde{Y}_c)$ is Stein analytic subset in S_c. We can take two neigborhoods V_c' and V_c in S_c such that $V_c' \supset V_c$ and $V_c' \cap \tau \circ \pi (\widetilde{Y}) = V_c \cap \tau \circ \pi (\widetilde{Y}) = \tau \circ \pi (\widetilde{Y}_c)$. Replacing V_c' and V_c by smaller neighorhoods, if necessary, we can choose a relatively compact neighborhood A of $(\tau \circ \pi)^{-1}(V_c) \cap \widetilde{Y}$ in $(\tau \circ \pi)^{-1}(V_c')$ such that $(\overline{W}_1 - W_2) \cap h^{-1}(A) \Subset W$, $\varphi(x) > \varepsilon$ on $x \in \partial W_1 \cap h^{-1}(A - \widetilde{Y})$ and $\varphi(x) < \varepsilon$ on $x \in \partial W_2 \cap h^{-1}(A - \widetilde{Y})$, where $h : L \to \widetilde{X}$ is the canonical projection. Here, there is a canonical holomorphic section $s(x)$ of the linear space L in the sense of Grauert which vanishes on $\widetilde{Y}$ and is nowhere zero on $\widetilde{X} - \widetilde{Y}$ (for the construction of $s(x)$, see [3], p. 353). Then we can chose a positive constant $\rho > 0$ such that $\rho s(\partial A - \widetilde{Y}) \cap \overline{W}_1 = \phi$. We put

$$\widetilde{U}_c = \{x \in A \cap (\tau \circ \pi)^{-1}(V_c) \,;\, \rho s(x) \in \overline{W}_2 \text{ or } \rho s(x) \in W_1 - \overline{W}_2 \text{ and } \varphi(\rho s(x)) < \varepsilon\}.$$

$\widetilde{U}_c$ is a neighborhood of $Y \cap (\tau \circ \pi)^{-1}(V_c)$ in $(\tau \circ \pi)^{-1}(V_c)$ with $\widetilde{U}_c \cap \widetilde{Y} = \widetilde{Y}_c$. It is clear that $\partial \widetilde{U}_c - \widetilde{Y}$ is strongly pseudoconvex. Next, we shall show that $\tau \circ \pi |_{\widetilde{U}_c} : \widetilde{U}_c \to S_c$ is 1-convex holomorphic map. We take a real valued C^∞-function $\mu : \{t \in \mathbf{R}; 0 \leqq t < \varepsilon\} \to \mathbf{R}$ such that $\mu(t) = 0$ for $0 \leqq t < \frac{\varepsilon}{2}$, $\mu(t)$, $\mu'(t)$ and $\mu''(t) > 0$ for $\varepsilon > t > \frac{\varepsilon}{2}$ and $\mu(t) \to \infty$ when $t \to \varepsilon$. We define a function Ψ on $\widetilde{U}_c$ by

$$\Psi = \begin{cases} \mu(\varphi(\rho s(x))) & \\ & \text{for } x \in \widetilde{U}_c \cap \{x \in A \cap (\tau \circ \pi)^{-1}(V_c) \,;\, \rho s(x) \in W_1 - \overline{W}_2 \text{ and } \varphi(\rho s(x)) < \varepsilon\} \\ 0 & \text{otherwise.} \end{cases}$$

From this, $\tau \circ \pi : \widetilde{U}_c \to S_c$ is a 1-convex holomorphic map with respect to Ψ. Hence by the result of Siu and Knorr-Schneider ([13], p. 212, Proposition 3.6 and [6], p. 244, Satz 3.4), we obtain a holomorphically convex neighborhood $\widetilde{U}_c$ of $\widetilde{Y}_c$ in $\widetilde{X}$ with $\widetilde{U}_c \cap \widetilde{Y} = \widetilde{Y}_c$ for every $c \in \mathbf{R}$. Since $\pi_*(\mathcal{O}_{\widetilde{U}_c}) = \mathcal{O}_{U_c}$, U_c is a holomorphically convex neighborhood of Y_c in X with $U_c \cap Y = Y_c$ for every $c \in \mathbf{R}$, where $U_c = \pi(\widetilde{U}_c)$. Further, by the construction of the neighborhoods $\widetilde{U}_c$ and W and of C^∞-psh function φ on W, no positive dimensional irreducible compact analytic subset of $\widetilde{U}_c$ intersects with $\widetilde{U}_c - \widetilde{Y}_c$ ([4], p. 273, proposition 3). Since there holds $\pi : \widetilde{X} \backslash \widetilde{Y} \rightrightarrows X \backslash Y$, no positive dimensional compact irreducible analytic subset of U_c intersects with $U_c - Y_c$ for every $c \in \mathbf{R}$. q. e. d.

REMARK 4. K. Takegoshi showed the same conclusion as Theorem 2.4 under the assumption that there hold (2) and (3) of Theorem 2.4 and the normal bundle N_Y of Y in the linear space is negative in the sense of Grauert [3], where Y is a closed submanifold of a complex manifold X ([16], Corollary of Theorem 2). On the other hand, to the contraction problem, we can apply a 1-convex holomorphic map with relation to Ohsawa's Theorem (see [9], [10]) ([9], §5, Proposition 5.1).

3. The Fundamental System of Neighborhoods of a Subspace in a Complex Space

(3.1) First, we shall show the existence of fundamental system of Stein neighborhoods for each level set Y_c in complex space X where Y is closed Stein subspace of X. For this proof we shall use Schneider's method [12].

THEOREM 3.1. *Let X be a complex space and let Y be a closed Stein subspace of X with a positive C^∞-spsh exhaustion function φ. Then there exists a Stein neighborhood U_c with $U_c \cap Y = Y_c$ for every $c \in R^+$, where $Y_c = \{y \in Y;\ \varphi(y) < c\}$.*

PROOF. Now by the monoidal transformation of X with center Y, we consider an effective cartier divisor $\widetilde{Y}$ on $\widetilde{X}$. Then there exists a proper holomorphic map $\pi: \widetilde{X} \to X$ such that $\pi: \widetilde{X}\backslash\widetilde{Y} \rightrightarrows X\backslash Y$. $\widetilde{Y}$ is given by $\{f_i=0\}$ on $\widetilde{U}_i$ where $\{\widetilde{U}_i\}$ is a suitable open covering of $\widetilde{Y}$ in $\widetilde{X}$ and each f_i is a holomorphic function on $\widetilde{U}_i$. Then $[\widetilde{Y}]$ has the transition function $g_{ij}=f_i \circ f_j^{-1}$ and an hermitian metric $\{h_i\}$ along the fibre by positive C^∞-functions h_i on $\widetilde{U}_i$ such that $h_j = |g_{ij}|^2 h_i$ on $\widetilde{U}_i \cap \widetilde{U}_j$. Hence we obtain $|f_i|^2 h_i = |f_j|^2 |g_{ij}|^2 h_i = |f_j|^2 h_j$ on $\widetilde{U}_i \cap \widetilde{U}_j$. This enables us to define a C^∞-function g on $\widetilde{X}$ by setting $g = |f_i|^2 h_i$ on $\widetilde{U}_i$. On the other hand, by Richberg's result ([11], Satz 3.3) φ is extended to a positive C^∞-spsh function Φ on a neighborhood Z of Y in X with $\Phi|_Y = \varphi$. We put $\widetilde{\Phi} = \Phi \circ \pi$ on $\pi^{-1}(Z)$. Then $\widetilde{\Phi}$ is a C^∞-spsh on $\pi^{-1}(Z) - \widetilde{Y}$. Again, we put $\widetilde{\Psi} = \widetilde{\psi}^{-1}$ where $\widetilde{\psi} = -\log(|f_i|^2 h_i e^{k\widetilde{\Phi}})$ on $\widetilde{U}_i$ for a natural number k. Then the Levi form $L(\widetilde{\Psi})$ of $\widetilde{\Psi}$ is as follows:

$$L(\widetilde{\Psi}) = \sum_{\alpha,\beta} \frac{\partial^2 \log h}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha d\bar{z}_\beta + 2\frac{1}{(\widetilde{\psi})^3} \left| \sum_\alpha \frac{\partial \widetilde{\psi}}{\partial z_\alpha} dz_\alpha \right|^2$$
$$+ k \sum_{\alpha,\beta} \frac{\partial^2 \widetilde{\Phi}}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha d\bar{z}_\beta. \qquad \text{on } \pi^{-1}(Z) - \widetilde{Y}.$$

where $(z_1, \cdots, z_n)$ are local coordinates for some open covering of X. Since $\widetilde{\Phi}$ is C^∞-spsh on $\pi^{-1}(Z) - \widetilde{Y}$, there exist a relatively compact neighborhood $\widetilde{U}_c$ of $\pi^{-1}(Y_c)$ in $\pi^{-1}(Z)$ and a natural number k_1 depending only on $\widetilde{U}_c$ such that $\widetilde{U}_c \cap \widetilde{Y} = \pi^{-1}(Y_c)$ and $L(\widetilde{\Psi}) > 0$ on $\widetilde{U}_c - \widetilde{Y}$ for every $k \geq k_1$. Then $\widetilde{\Psi}$ is a C^∞-spsh on $\widetilde{U}_c - \widetilde{Y}$ and vanishes only on $\widetilde{Y}$ since $f_i = 0$ on $\widetilde{U}_i \cap \widetilde{Y}$. Since $\pi: \widetilde{X}\backslash\widetilde{Y} \rightrightarrows X\backslash Y$, we can take a C^∞-psh function Ψ with $\Psi \circ \pi = \widetilde{\Psi}$. Hence we can see that Ψ vanishes only on Y and C^∞-spsh on $U_c - Y$, where $U_c = \pi(\widetilde{U}_c)$. For the two functions Φ and Ψ, we set $\rho =$

$\Phi+\frac{C}{\varepsilon}\Psi$, where ε is chosen dependently on c and $\varepsilon>0$. The function ρ is clearly C^∞-spsh on U_c. We put $U_{c,\varepsilon}=\{x\in U_c;\ \rho(x)<c\}$, then $U_{c,\varepsilon}$ is the relatively compact subset of U_c if we take ε sufficiently small (see the proof of Lemma. 2.0). Hence $U_{c,\varepsilon}$ is a Stein subspace with a C^∞-spsh exhaustion function $(c-\rho)^{-1}$([8], p. 195, Theorem II) and $U_{c,\varepsilon}\cap Y=Y_c$ since Ψ vanishes only on Y. Moreover, for a sufficiently small ε $U_{c,\varepsilon}$ is arbitrarily small. Here we denote by U_c instead of $U_{c,\varepsilon}$. This completes the proof of our theorem. q. e. d.

REMARK 5. Let X be a complex manifold and let Y be a closed Stein submanifold. Then by Remark 2, Lemma 1.1 and Richberg's result ([11], Satz 3.3), we have the same conclusion as Theorem 3.1.

(3.2) Next, we want to consider the problem as same as Theorem 3.1 for holomorpically convex subspace in complex space. But in this case, unfortunately we can not use Richberg's result for C^∞-spsh function on subspace. Therefore we shall consider the problem under some conditions for holomorphically convex subspace in complex space.

PROPOSITION 3.2. *Let X be a complex space and let Y be a closed subspace of X. Let $[Y]$ be the line bundle corresponding to the effective cartier divisor Y. Assume that the following conditions are satisfied :*

(1) *Y is holomorpically convex,*

(2) *the restriction $[\widetilde{Y}]|_{\widetilde{Y}}$ of $[\widetilde{Y}]$ to $\widetilde{Y}$ is negative,*

(3) *there exist a positive constant c_o and a neighborhood V_c of Y_c in X such that $H^1(V_c,\mathscr{I}_{Y_c})=0$ for every c with $c_o>c>0$, where $\mathscr{I}_{Y_c}$ is the ideal sheaf of Y_c.*

Then for every c where $c_o>c>0$, there exists a sufficiently small weakly 1-complete neighborhood U_c of Y_c in X with $U_c\cap Y=Y_c$.

PROOF. Let π: $\widetilde{X}\to X$ be the monoidal transformation of X with center Y. Since Y is holomorphically convex and π is proper, we can consider that $\widetilde{Y}$ is a weakly 1-complete effective cartier divisor. Then from Lemma 1.2, there exist a neighborhood W_c of Y_c in X with $W_c\cap Y=Y_c$ and a C^∞-psh function ψ on W_c which is C^∞-spsh on W_c-Y_c and vanishes only on Y_c. On the other hand, since Y is holomorphically convex, by Remmert's reduction theorem, there exist $f_1,\cdots,f_N\in H^0(Y,\mathcal{O}_Y)$ which define a proper holomorphic map $(f_1,\cdots,f_N)$: $Y\to \boldsymbol{C}^N$ such that $Y_c=\{y\in Y;\ \sum_{i=1}^N|f_i(y)|^2<c\}$.

From the exact sequence

$$0\longrightarrow\mathscr{I}_{Y_c}\longrightarrow\mathcal{O}_{V_c}\to\mathcal{O}_{Y_c}\longrightarrow 0\quad(c_o>c>0),$$

we obtain a surjective map

$$H^0(V_c,\mathcal{O}_{V_c})\longrightarrow H^0(Y_c,\mathcal{O}_{Y_c}).$$

Hence f_i is extended to $\widetilde{f}_i$ on V_c such that $\widetilde{f}_i|_{Y_c} = f_i$ $(i=1,\cdots,N)$ for every c with $c_0>c>0$. We put $\widetilde{\varphi} = \sum_{i=1}^{N} |\widetilde{f}_i|^2$ on V_c. Then $\widetilde{\varphi}$ is a C^∞-psh function on V_c. For $\varepsilon>0$ we set

$$(*) \qquad \Phi = \widetilde{\varphi} + \frac{c}{\varepsilon}\psi \quad \text{on } V_c \cap W_c.$$

Then Φ is a C^∞-psh function on $V_c \cap W_c$. $U_c = \{x \in V_c \cap W_c;\ \Phi(x) < c\}$ is relatively compact in $V_c \cap W_c$ for sufficiently small $\varepsilon>0$ (see the proof of Lemma 2.0) and $U_c \cap Y = Y_c$ since $\psi=0$ on Y_c. Hence, U_c is a weakly 1-complete neighborhood of Y_c with an exhaustion function $(c-\Phi)^{-1}$. Moreover, for a sufficiently small ε, U_c is arbitrarily small neighborhood of Y_c in X. We conclude the proof of this proposition. q. e. d.

THEOREM 3.3. *Under the above conditions* (1)–(3) *of Proposition* 3.2, *assume further the following condition* (4):

(4) *X is a normal complex space.*

Then there exists a holomorphically convex neighborhood U_c of Y_c in X such that $U_c \cap Y = Y_c$ and no positive dimensional compact irreducible subvariety of U_c intersects with $U_c - Y_c$ for every $c \in \boldsymbol{R}$ with $c_0>c>0$.

PROOF. Now, for fixed c and c' with $c_0>c'>c>0$ we put $\widetilde{Y}_c = \pi^{-1}(Y_c)$ and $\widetilde{U}_{c'} = \pi^{-1}(U_{c'})$. By Proposition 3.2, since $U_{c'}$ is weakly 1-complete in X, $\widetilde{U}_{c'}$ is also weakly 1-complete neighborhood of $\widetilde{Y}_c$ in $\widetilde{X}$. On the other hand, from the similar construction of an exhaustion function on $\widetilde{U}_{c'}$ as (*) of Proposition 3.2, we can find a weakly 1-complete neighborhood $\widetilde{U}_c$ of $\widetilde{Y}_c$ in $\widetilde{U}_{c'}$ such that $\widetilde{U}_c \cap \widetilde{Y} = \widetilde{Y}_c$, where $\widetilde{U}_c$ is defined from the exhaustion function of weakly 1-complete $\widetilde{U}_{c'}$. Following, if necessary, we replace $\widetilde{U}_{c'}$ by a smaller neighborhood without a particular mention of it. First we shall prove that for every point $\widetilde{x}_0 \in \partial\widetilde{U}_c - \partial\widetilde{U}_c \cap \widetilde{Y}$ there exists an element $\widetilde{g}_{\widetilde{x}_0} \in H^0(\widetilde{U}_c, \mathcal{O}_{\widetilde{U}_c})$ such that $\lim_{\substack{\widetilde{x}\to\widetilde{x}_0 \\ x\in\widetilde{U}_c}} \widetilde{g}_{\widetilde{x}_0}(\widetilde{x}) = \infty$. By applying Fujiki's Lemma to $\widetilde{Y}_{c'}$, we have a neighborhood $\widetilde{V}_{c'}$ of $\widetilde{Y}_{c'}$ in $\widetilde{X}$ with $\widetilde{V}_{c'} \cap \widetilde{Y} = \widetilde{Y}_{c'}$ such that $[\widetilde{Y}]$ is negative on $\widetilde{V}_{c'}$ and $\widetilde{V}_{c'}$ contains $\widetilde{U}_{c'}$. The point $\widetilde{x}_0$ has a property that an exhaustion function $\widetilde{\Phi}$ of $\widetilde{U}_{c'}$ is C^∞-spsh at $\widetilde{x}_0$ (see the proof of Proposition 3.2). Then by Satz 1.4 in [3], there exist a neighborhood $\widetilde{U}$ of $\widetilde{x}_0$ in $\widetilde{U}_{c'}$ and effective cartier divisor Z' of $\widetilde{U}$ which is Stein such that $Z' \cap \widetilde{U}_c = \{\widetilde{x}_0\}$. We take $c_1(>c)$ sufficiently near c so that $Z = Z' \cap \widetilde{U}_{c_1}$ is an analytic subset of $\widetilde{U}_{c_1}$and $Y \cap Z = \phi$. Here, $\widetilde{U}_c$ is a weakly 1-complete neighborhood with an exhaustion function $\widetilde{\Phi}_{c_1} = (c_1 - \widetilde{\Phi})^{-1}$ and we denote by $\widetilde{\Phi}_Z$ the restriction of $\widetilde{\Phi}_{c_1}$ to Z. Then, since $\widetilde{\Phi}_Z$ is C^∞-spsh function on Z and $\{w \in Z;\ \widetilde{\Phi}_Z(w) < d\} \Subset Z$ for every $d > (c_1-c)^{-1}$, Z is Stein. We put $B = [Z]$ as the corresponding line bundle. We consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{U}_{c'}}(\widetilde{L}_{c'}^{*\otimes m}) \longrightarrow \mathcal{O}_{\widetilde{U}_{c'}}(\widetilde{L}_{c'}^{*\otimes m} \otimes B) \longrightarrow \mathcal{O}_Z(B) \longrightarrow 0,$$

where $\widetilde{L} = [\widetilde{Y}]$, $\widetilde{L}_{c'} = [\widetilde{Y}]|_{\widetilde{U}_{c'}}$, m is a positive integer and $\widetilde{L}^*_{c'}$ is the dual bundle of $\widetilde{L}_{c'}$. From this, we obtain a cohomology exact sequence

$$\longrightarrow H^0(\widetilde{U}_{c'}, \mathcal{O}_{U_{c'}}(\widetilde{L}_{c'}^{*\otimes m} \otimes B)) \xrightarrow{f} H^0(Z, \mathcal{O}_Z(B))$$

$$\longrightarrow H^1(\widetilde{U}_{c'}, \mathcal{O}_{\widetilde{U}_{c'}}(\widetilde{L}_{c'}^{*\otimes m})) \longrightarrow .$$

By applying Theorem F to $\widetilde{L}_{c'}$, if we take m sufficiently large we have $H^1(\widetilde{U}_{c'}, \mathcal{O}_{\widetilde{U}_{c'}}(\widetilde{L}_{c'}^{*\otimes m}))=0$. Hence the map f is surjective. Since Z is Stein, by Cartan's Theorem A, we can find an element $s \in H^0(Z, \mathcal{O}_Z(B))$ such that $s(\widetilde{x}_0) \neq 0$. Since f is surjective, there exists an element $\widetilde{s}$ of $H^0(\widetilde{U}_{c'}, \mathcal{O}_{\widetilde{U}_{c'}}(\widetilde{L}_{c'}^{*\otimes m} \otimes B))$ such that $f(\widetilde{s}) = s$. By virtue of the natural isomorphism, we have $\mathcal{O}_{\widetilde{U}_{c'}}(\widetilde{L}_{c'}^{*\otimes m}) = \mathcal{I}^m_{\widetilde{Y}_{c'}}$ for every $m \geqq 1$, where $\mathcal{I}_{\widetilde{Y}_{c'}}$ is the ideal sheaf of $\widetilde{Y}_{c'}$ in $\widetilde{U}_{c'}$. From this isomorphism, $\widetilde{s}$ can be regarded as an element $\widetilde{s}'$ of $H^0(\widetilde{U}_{c'}, \mathcal{O}_{Y_{c'}}(B) \otimes \mathcal{I}^m_{\widetilde{Y}_{c'}})$. Then this $\widetilde{s}'$ gives a meromorphic function $\widetilde{g}'$ on $\widetilde{U}_{c'}$ which has a pole on Z and is holomorphic outside Z. Since $Z \cap \widetilde{U}_c = \{\widetilde{x}_0\}$, $\widetilde{g} = \widetilde{g}'|_{\widetilde{U}_c}$ is the desired function (this proof is due to Fujiki [2], Lemma 7). We put $U_c = \pi(\widetilde{U}_c)$, then $U_c \cap Y = Y_c$. Now, since X is normal, we can take $\pi_*(\mathcal{O}_{\widetilde{U}_c}) = \mathcal{O}_{U_c}$. Hence for every point $x_0 \in \partial U_c - \partial U_c \cap Y$, there exists an element $g_{x_0} \in H^0(U_c, \mathcal{O}_{U_c})$ such that $\lim\limits_{\substack{x \to x_0 \\ x \in U_c}} g_{x_0}(x) = \infty$. On the other hand, since Y is holomorphically convex and since the condition (3) is satisfied, we can find also an element g_{x_1} with the similar property as the above for every point x_1 on $\partial U_c \cap Y$. Hence U_c is a holomorphically convex neighborhood of Y_c in X so that $U_c \cap Y = Y_c$. Further, from the construction of neighborhood U_c, there exists a C^∞-psh function on U_c which is C^∞-spsh on $U_c - Y_c$. Then no positive dimensional irreducible compact subvariety of U_c intersects with $U_c - Y_c$ ([4], p. 273, Proposition 3). q. e. d.

REMARK 6. In the case of holomorphically convex subspace Y which is an effective cartier divisor, Fujiki showed that for every $c \in \boldsymbol{R}$ each level set Y_c has the same conclusion as Theorem 3.3 under the assumption that the restriction $[Y]|_Y$ of the corresponding line bundle to Y is negative and $H^1(Y, \mathcal{O}(([Y]|_Y{}^{*\otimes\mu}))=0$ for every $\mu \geqq 1$, where $([Y]|_Y)^*$ is the dual bundle of $[Y]|_Y$ ([2], p.483, Theorem 1, pp. 491–493 and p. 493, Remark 1).

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